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Effective thermal conductivity of nonlinear composite media with contact resistance

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Abstract—A perturbation expansion method is used to study the effective thermal conductivity of nonlinear composite media with contact resistance. We overcome difficulties caused by the temperature discontinuity and temperature gradient singularity on surfaces between different phases where thermal contact resistance exists. A general definition of the effective thermal conductivity is proposed. Using the definition, we derive formulae for the effective linear and nonlinear thermal conductivities of composite media with low concentrations of cylindrical inclusions. Effects of thermal contact resistance on the effective thermal conductivities are discussed in detail. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Thermal transport properties of composite media remain an active area of research because of their increasing applications in engineering fields [1–5]. Using the Rayleigh method [6], Gu and Tao [4] investigated the effective thermal conductivity of composite media containing spheres with contact resistance. In order to calculate the effective conductivity for a dry spent nuclear fuel assembly which consists of fuel rods in a regular pattern, Manteufel and Todreas [5] derived a set of analytical formulae and took into account the contact resistance to a certain degree. Both Gu and Manteufel and many other researchers assumed that heat conduction in the two phases of composite media obey the linear Fourier law.

The physical properties of nonlinear composite media have attracted much interest in the past few years [7–10]. Much work has been done on the nonlinear optical and electrical susceptibilities of composite media with cylindrical or spherical inclusions periodically embedded in a matrix. In heat transfer, nonlinearity also plays an important role when the applied temperature gradient is high or the thermal conductivity varies greatly with temperature [11].

This paper combines and extends the earlier results of nonlinearity of composites and effects of contact resistance by using a perturbation expansion method, concentrating on deriving the effective thermal conductivity and analyzing the influence of contact resistance [1, 4, 5, 12, 13] which proves to be difficult in the case of nonlinearity.

The organization of this paper is as follows. In Section 2, we review briefly the perturbation expansion method and establish general expansions for field equations and boundary conditions of nonlinear composite media. As an example, Section 3 deals with a simple two-dimensional example of a cylindrical inclusion embedded in a matrix. We derive the analytical expansions for the temperature in composite media in the presence of a uniform external temperature field. In Section 4, we generalize the method to calculate the effective thermal conductivities of nonlinear composites and obtain formulae for the effective conductivities at low inclusion concentrations. Section 5 presents conclusions.

2. PERTURBATION EXPANSION METHOD

For isotropic nonlinear composite media, the heat flow density q^{m} in the matrix region and q^{i} in the inclusion region are supposed to be related to the local temperature fields ∇T^{m} and ∇T^{i}, respectively, by the following nonlinear constitutive equations:

$$q^{\mathrm{m}} = -\lambda_{\mathrm{m}}\nabla T^{\mathrm{m}} - \mu_{\mathrm{m}}|\nabla T^{\mathrm{m}}|^2\nabla T^{\mathrm{m}} \quad \text{in } \Omega_{\mathrm{m}} \tag{1}$$

and

$$q^{\mathrm{i}} = -\lambda_{\mathrm{i}}\nabla T^{\mathrm{i}} - \mu_{\mathrm{i}}|\nabla T^{\mathrm{i}}|^2\nabla T^{\mathrm{i}} \quad \text{in } \Omega_{\mathrm{i}} \tag{2}$$

where λ_{m}, λ_{i} and μ_{m}, μ_{i} are called the first and third thermal conductivity coefficients, respectively, and Ω_{m} and Ω_{i} stand for the matrix region and inclusion region. In what follows, subscript (or superscript) m

NOMENCLATURE

a	radius of inclusion
B, b, c, D, L	coefficients in formulae
BI	Biot number
F	defined by equations (31)–(36)
f	volume concentration of particle
G	defined by equation (11)
h	film coefficient
k	ratio of λ_i/λ_m
k_u	ratio of μ_i/μ_m
n	unit outward normal on surface of inclusion
q	heat flow vector
r, ϕ	polar coordinates
T	temperature
T_e	applied temperature gradient.

Greek symbols

Ω	domain
α	defined by equation (10)
ε	perturbation parameter
λ	first-order conductivity
μ	third-order conductivity.

Subscripts or superscripts

a, b, c	cases of discussion about integrals over interface in Section 4
i	inclusion
m	matrix
p	= m, i
*	effective quantity.

is used to denote the quantities in the matrix and subscript (or superscript) i is used to denote the quantities in the inclusion.

In the steady state, heat flow density satisfies the following equations:

$$\nabla q^m = 0 \quad \text{in } \Omega_m \tag{3}$$

and

$$\nabla q^i = 0 \quad \text{in } \Omega_i \tag{4}$$

The boundary condition for continuity of heat flow is:

$$nq^m = nq^i \quad \text{on } \partial\Omega_i \tag{5}$$

where n is unit outward normal at the surface of an inclusion.

For composites with thermal contact resistance, the boundary condition for temperature is:

$$-\lambda_i \frac{\partial T^i}{\partial n} = h(T^i - T^m) \quad \text{on } \partial\Omega_i \tag{6}$$

where h is the film coefficient. A smaller h indicates a greater contact resistance. If two phases of composite media have an ideal contact, i.e. $h = \infty$, equation (6) changes into

$$T^i = T^m \quad \text{on } \partial\Omega_i. \tag{7}$$

We choose ε, equal to either μ_m or μ_i, as the expansion parameter. In case of weak nonlinearity, the convergence region is decided by the condition [10] $|q_{\text{nonlinear}}| < |q_{\text{linear}}|$, which leads to $\mu|\nabla T|^2/\lambda < 1$. The expansions for temperature read:

$$T^m = T_0^m + \varepsilon T_1^m + \varepsilon^2 T_2^m + ,\ldots, \quad \text{in } \Omega_m \tag{8}$$

$$T^i = T_0^i + \varepsilon T_1^i + \varepsilon^2 T_2^i + ,\ldots, \quad \text{in } \Omega_i. \tag{9}$$

It is convenient to define the following quantities:

$$\alpha_p = \mu_p/\varepsilon \quad p = \text{m}, \text{i} \tag{10}$$

$$G^p = |\nabla T^p|^2 \quad p = \text{m}, \text{i}. \tag{11}$$

We write G^i, q^p as expansions in ε, too:

$$\begin{aligned} G^p &= G_0^p + \varepsilon G_1^p + \varepsilon^2 G_2^p + ,\ldots, \\ &= (\nabla T_0^p)^2 + 2\varepsilon(\nabla T_0^p \nabla T_1^p) \\ &\quad + \varepsilon^2[(\nabla T_1^p)^2 + 2\nabla T_0^p \nabla T_2^p] + ,\ldots, \quad p = \text{m}, \text{i} \end{aligned} \tag{12}$$

$$\begin{aligned} q^p &= q_0^p + \varepsilon q_1^p + \varepsilon^2 q_2^p + ,\ldots, \\ &= -\lambda_p \nabla T_0^p - \varepsilon[\lambda_p \nabla T_1^p \\ &\quad + \alpha_p G_0^p \nabla T_0^p] \\ &\quad - \varepsilon^2[\lambda_p \nabla T_2^p + \alpha_p(G_1^p \nabla T_0^p \\ &\quad + G_0^p \nabla T_1^p)] - ,\ldots, \quad p = \text{m}, \text{i}. \end{aligned} \tag{13}$$

Application of equation (13) to equations (3) and (4) yields perturbation equations for each order of ε:

$$\begin{aligned} &\lambda_p \nabla^2 T_0^p = 0 \quad \lambda_p \nabla^2 T_1^p + \alpha_p(\nabla G_0^p \nabla T_0^p) = 0 \\ &\lambda_p \nabla^2 T_2^p + \alpha_p(\nabla G_1^p \nabla T_0^p + \nabla G_0^p \nabla T_1^p + G_0^p \nabla^2 T_1^p) = 0 \ldots \\ &\qquad \text{in } \Omega_p \quad p = \text{m}, \text{i}. \end{aligned} \tag{14}$$

Introduction of equations (8) and (9) into equation (6) and equation (13) into equation (5) gives the boundary conditions corresponding to each order:

$$-\lambda_i \frac{\partial T_j^i}{\partial n} = h(T_j^i - T_j^m) \quad j = 0, 1, 2, \ldots, \quad \text{on } \partial\Omega_i \tag{15}$$

and

$$nq_j^i = nq_j^m \quad j = 0, 1, 2, \ldots, \quad \text{on } \partial\Omega_i. \tag{16}$$

In addition, the temperature should remain finite at the origin and the temperature gradient ∇T should coincide with the external temperature field at infinity.

3. TEMPERATURE FIELDS IN COMPOSITE MEDIA

As the first step, we consider a simple case in which a cylindrical inclusion of radius a is embedded in a matrix and subject to a uniform external temperature gradient field T_e along the x direction. Analytical solutions of this dilute system can be used to derive various formulae of nonlinear EMA (effective medium approximation) for nonlinear composites with a low particle concentration. Using equations (14)–(16), we get the following equations and corresponding boundary conditions for each order of the temperature fields.

For the zeroth order, the governing equations are just the same as those in linear systems:

$$\nabla^2 T_0^{\mathrm{m}} = 0 \quad \text{in } \Omega_{\mathrm{m}} \tag{17}$$

$$\nabla^2 T_0^{\mathrm{i}} = 0 \quad \text{in } \Omega_{\mathrm{i}} \tag{18}$$

subject to boundary conditions

$$-\lambda_{\mathrm{i}}\nabla_r T_0^{\mathrm{i}} = h(T_0^{\mathrm{i}} - T_0^{\mathrm{m}}) \quad \text{on } \partial\Omega_{\mathrm{i}} \tag{19}$$

$$\lambda_{\mathrm{i}}\nabla_r T_0^{\mathrm{i}} = \lambda_{\mathrm{m}}\nabla_r T_0^{\mathrm{m}} \quad \text{on } \partial\Omega_i. \tag{20}$$

For the first-order,

$$\nabla^2 T_1^{\mathrm{m}} = -\frac{\alpha_{\mathrm{m}}}{\lambda_{\mathrm{m}}}\nabla G_0^{\mathrm{m}}\nabla T_0^{\mathrm{m}} \quad \text{in } \Omega_{\mathrm{m}} \tag{21}$$

$$\nabla^2 T_1^{\mathrm{i}} = -\frac{\alpha_{\mathrm{i}}}{\lambda_{\mathrm{i}}}\nabla G_0^{\mathrm{i}}\nabla T_0^{\mathrm{i}} \quad \text{in } \Omega_{\mathrm{i}} \tag{22}$$

subject to

$$-\lambda_{\mathrm{i}}\nabla_r T_1^{\mathrm{i}} = h(T_1^{\mathrm{i}} - T_1^{\mathrm{m}}) \quad \text{on } \partial\Omega_{\mathrm{i}} \tag{23}$$

$$\lambda_{\mathrm{i}}\nabla_r T_1^{\mathrm{i}} + \alpha_{\mathrm{i}} G_0^{\mathrm{i}}\nabla_r T_0^{\mathrm{i}} = \lambda_{\mathrm{m}}\nabla_r T_1^{\mathrm{i}} + \alpha_{\mathrm{m}} G_0^{\mathrm{m}}\nabla_r T_0^{\mathrm{m}} \quad \text{on } \partial\Omega_{\mathrm{i}}. \tag{24}$$

For the second-order,

$$\nabla^2 T_2^{\mathrm{m}} = -\frac{\alpha_{\mathrm{m}}}{\lambda_{\mathrm{m}}}[\nabla G_1^{\mathrm{m}}\nabla T_0^{\mathrm{m}} + \nabla G_0^{\mathrm{m}}\nabla T_1^{\mathrm{m}} + G_0^{\mathrm{m}}\nabla^2 T_1^{\mathrm{m}}] \quad \text{in } \Omega_{\mathrm{m}} \tag{25}$$

$$\nabla^2 T_2^{\mathrm{i}} = -\frac{\alpha_{\mathrm{i}}}{\lambda_{\mathrm{i}}}[\nabla G_1^{\mathrm{i}}\nabla T_0^{\mathrm{i}} + \nabla G_0^{\mathrm{i}}\nabla T_1^{\mathrm{i}} + G_0^{\mathrm{i}}\nabla^2 T_1^{\mathrm{i}}] \quad \text{in } \Omega_{\mathrm{i}} \tag{26}$$

subject to

$$-\lambda_{\mathrm{i}}\nabla_r T_2^{\mathrm{i}} = h(T_2^{\mathrm{i}} - T_2^{\mathrm{m}}) \quad \text{on } \partial\Omega_{\mathrm{i}} \tag{27}$$

$$\lambda_{\mathrm{i}}\nabla_r T_2^{\mathrm{i}} + \alpha_{\mathrm{i}}(G_1^{\mathrm{i}}\nabla_r T_0^{\mathrm{i}} + G_0^{\mathrm{i}}\nabla_r T_1^{\mathrm{i}}) = \lambda_{\mathrm{m}}\nabla_r T_2^{\mathrm{m}} + \alpha_{\mathrm{m}}(G_1^{\mathrm{m}}\nabla_r T_0^{\mathrm{m}} + G_0^{\mathrm{m}}\nabla_r T_1^{\mathrm{m}}) \text{on} \quad \partial\Omega_{\mathrm{i}}. \tag{28}$$

In addition, we must impose another two conditions: T^{i} must remain finite for $r = 0$ and T^{m} must satisfy

$$\frac{\partial T_0^{\mathrm{m}}}{\partial x} = T_e \quad \text{for } r \to \infty \tag{29}$$

and

$$\frac{\partial T_j^{\mathrm{m}}}{\partial x} = 0 \quad j = 0, 1, 2, \ldots, \quad \text{for } r \to \infty \tag{30}$$

These boundary value problems can be solved order by order. For the first- and higher-order, one needs to solve nonhomogeneous linear differential equations such as equations (21), (22), (25) and (26), etc. The right-hand side of the jth order equations is decided by the solutions to all former-order equations. Omitting the tedious derivations, we directly give the result:

$$T_0^{\mathrm{m}} = F_0^{\mathrm{m}} T_e = (r + br^{-1})\cos\varphi T_e \tag{31}$$

$$T_0^{\mathrm{i}} = F_0^{\mathrm{i}} T_e = cr\cos\varphi T_e \tag{32}$$

$$\begin{aligned} T_1^{\mathrm{m}} &= F_1^{\mathrm{m}} T_e^3 \\ &= \{[b_1 r^{-1} + \alpha_{\mathrm{m}} b^2 r^{-3}/\lambda_{\mathrm{m}} - \alpha_{\mathrm{m}} b^3 r^{-5}/(6\lambda_{\mathrm{m}})]\cos\varphi \\ &\quad + [b_2 r^{-3} + \alpha_{\mathrm{m}} b r^{-1}/(2\lambda_{\mathrm{m}})]\cos 3\varphi\} T_e^3 \end{aligned} \tag{33}$$

$$T_1^{\mathrm{i}} = F_1^{\mathrm{i}} T_e^3 = [b_3 r\cos\varphi + b_4 r^3\cos 3\varphi] T_e^3 \tag{34}$$

$$\begin{aligned} T_2^{\mathrm{m}} = F_2^{\mathrm{m}} T_e^5 &= [(c_1 r^{-1} + b_5 r^{-3} + b_6 r^{-5} + b_7 r^{-7} + b_8 r^{-9})\cos\varphi \\ &\quad + (c_2 r^{-3} + b_9 r^{-1} + b_{10} r^{-5} + b_{11} r^{-7} + b_{12} r^{-9})\cos 3\varphi + (c_3 r^{-5} + b_{13} r^{-1} \\ &\quad + b_{14} r^{-3})\cos 5\varphi] T_e^5 \end{aligned} \tag{35}$$

$$T_2^{\mathrm{i}} = F_2^{\mathrm{i}} T_e^5 = [(c_4 r + b_{15} r^3)\cos\varphi + c_5 r^3\cos 3\varphi + c_6 r^5\cos 5\varphi] T_e^5 \tag{36}$$

where

$b = Ba^2$

$c = 2/(1 + k + k/BI)$

$k = \lambda_{\mathrm{i}}/\lambda_{\mathrm{m}}$

$B = 1 - kc$

$BI = ha/\lambda_{\mathrm{m}}$ (Biot number)

$$\begin{aligned} b_1 = B_1 a^2 &= a^2[(1 - 2B - B^2 - B^3/6)\alpha_{\mathrm{m}} + (-B^2 + B^3/6)\alpha_{\mathrm{m}} k(1 + k/BI) - \alpha_{\mathrm{i}} c^3] \\ &\quad \times (1 + k/BI)/[\lambda_{\mathrm{m}}(1 + k + k/BI)] \end{aligned}$$

$$\begin{aligned} b_2 &= B_2 a^4 \\ &= a^4\{-B/2 + B^2/3 - Bk/[2(1 + 3k/BI)]\} \\ &\quad \times \alpha_{\mathrm{m}}(1 + 3k/BI)/[\lambda_{\mathrm{m}}(1 + k + 3k/BI)] \end{aligned}$$

$$b_3 = B_3 = [(1-2B-B^3/3)\alpha_m - \alpha_i c^3]/[\lambda_m(1+k+k/BI)]$$
$$b_4 = B_4 a^{-2} = a^{-2}B^2\alpha_m/[3\lambda_m(1+k+3k/BI)]$$
$$b_5 = B_5 a^4 = a^4[-5B^2\alpha_m^2/(2\lambda_m)+2BB_1\alpha_m]/\lambda_m$$
$$b_6 = B_6 a^6 = a^6[17B^3\alpha_m^2/(6\lambda_m)-B^2B_1\alpha_m/2 + BB_2\alpha_m]/\lambda_m$$
$$b_7 = B_7 a^8 = a^8[-19B^4\alpha_m^2/(12\lambda_m) - B^2B_2\alpha_m/2]/\lambda_m$$
$$b_8 = B_8 a^{10} = a^{10}[7B^5\alpha_m^2/(30\lambda_m^2)]$$
$$b_9 = B_9 a^2 = a^2[-B\alpha_m^2/\lambda_m + B_1\alpha_m/2]/\lambda_m$$
$$b_{10} = B_{10}a^6 = a^6[2B^3\alpha_m^2/\lambda_m + 3BB_2\alpha_m]/\lambda_m$$
$$b_{11} = B_{11}a^8 = a^8[-11B^4\alpha_m^2/(30\lambda_m) - 3B^2B_2\alpha_m/5]/\lambda_m$$
$$b_{12} = B_{12}a^{10} = a^{10}[B^5\alpha_m^2/(36\lambda_m^2)]$$
$$b_{13} = B_{13}a^2 = a^2[B\alpha_m^2/(4\lambda_m^2)]$$
$$b_{14} = B_{14}a^4 = a^4[3B^2\alpha_m^2/(4\lambda_m)+3B_2\alpha_m/2]/\lambda_m$$
$$b_{15} = B_{15}a^{-2} = a^{-2}[-3B_4c^2\alpha_i/(2\lambda_i)]$$

$$c_1 = C_1a^2 = a^2[L_1\lambda_i - D_1(1+k/BI)]/[\lambda_m(1+k+k/BI)]$$
$$c_2 = C_2a^4 = a^4[L_3\lambda_i - D_3(1+3k/BI)/3]/[\lambda_m(1+k+3k/BI)]$$
$$c_3 = C_3a^6 = a^6[L_5\lambda_i - D_5(1+5k/BI)/5]/[\lambda_m(1+k+5k/BI)]$$
$$c_4 = C_4 = [-L_1\lambda_m - D_1]/[\lambda_m(1+k+k/BI)]$$
$$c_5 = C_5a^{-2} = a^{-2}[-L_3\lambda_m - D_3/3]/[\lambda_m(1+k+3k/BI)]$$
$$c_6 = C_6a^{-4} = a^{-4}[-L_5\lambda_m - D_5/5]/[\lambda_m(1+k+5k/BI)]$$

$$L_1 = -(B_5+B_6+B_7+B_8)+B_{15}(1+3k/BI)$$
$$L_3 = -(B_9+B_{10}+B_{11}+B_{12})$$
$$L_5 = -(B_{13}+B_{14})$$

$$D_1 = (3B_5+5B_6+7B_7+9B_8)\lambda_m + \alpha_m(2B_1-4BB_1+3B_1B^2-6B_2B + 3B_2B^2)+\alpha_m^2(-B/2+6B^2 - 13B^3+34B^4/3+13B^5/6)/\lambda_m + \alpha_i(-3c^2B_4/2+3c^2B_3)$$

$$D_3 = (B_9+5B_{10}+7B_{11}+9B_{12})\lambda_m + \alpha_m(B_1-2B_1B+6B_2-6B_2B+6B_2B^2) + \alpha_m^2(B+B^2-11B^3/2+8B^4/3 - B^5/3)/\lambda_m + 6\alpha_i c^2 B_4$$
$$D_5 = (B_{13}+3B_{14})\lambda_m + \alpha_m(3B_2-6B_2B)+\alpha_m^2(B-B^2-B^3/2)/\lambda_m.$$

For the same BI, coefficients B, c, B_1–B_{15}, C_1–C_6, D_1–D_5, and L_1–L_5 are independent of radius a of the inclusion.

4. EFFECTIVE THERMAL CONDUCTIVITIES

In this section, we derive formulae for calculating effective thermal conductivities of nonlinear composite media with contact resistance. The average value of heat flow over a cell can be expressed as follows:

$$\langle q\rangle = \langle -\lambda\nabla T - \mu|\nabla T|^2\nabla T\rangle. \tag{37}$$

Considering the composite as an equivalent homogenous medium with effective thermal conductivities and taking into account the macroscopic behavior under an external homogeneous temperature gradient T_e, then

$$\langle q\rangle = -\lambda^*\langle\nabla T\rangle - \mu^*\langle|\nabla T|^2\nabla T\rangle - ,\ldots, \tag{38}$$

where λ^* and μ^* are the first- and third-order effective thermal conductivities, respectively.

Combining equations (37) and (38), we establish a general definition for calculating the effective thermal conductivities:

$$\langle\lambda\nabla T\rangle + \langle\mu|\nabla T|^2\nabla T\rangle = \lambda^*\langle\nabla T\rangle + \mu^*\langle|\nabla T|^2\nabla T\rangle,\ldots. \tag{39}$$

The preceding calculations have revealed some regularity: the jth terms of T_j^m and T_j^i, are proportional to T_e^{2j+1}, that is:

$$T_j^p \propto T_e^{2j+1}, \quad p = \mathrm{m,i}, \quad j = 0,1,2,\ldots. \tag{40}$$

Equation (39) must be satisfied for an arbitrary external temperature field. So the coefficients of different powers of T_e must be zero. Therefore, we have sufficient equations to determine the effective conductivities of the nonlinear composite media. In this paper we only write the first two equations:

$$\langle\lambda\nabla T\rangle = \lambda^*\langle\nabla T\rangle \tag{41}$$

$$\varepsilon\langle\lambda\nabla T_1\rangle + \langle\mu|\nabla T_0|^2\nabla T_0\rangle = \lambda^*\varepsilon\langle\nabla T_1\rangle + \mu^*\langle|\nabla T_0|^2\nabla T_0\rangle. \tag{42}$$

The effective conductivities on the plane normal to cylinder axes are isotropic. So calculating effective conductivities along the x axis is enough.

$$\lambda^* = \frac{\langle\lambda\nabla_x T_0\rangle}{\langle\nabla_x T_0\rangle} \tag{43}$$

$$\mu^* = \frac{\varepsilon\langle\lambda\nabla_x T_1\rangle + \langle\mu(\nabla T_0)^2\nabla_x T_0\rangle - \lambda^*\varepsilon\langle\nabla_x T_1\rangle}{\langle(\nabla T_0)^2\nabla_x T_0\rangle}. \tag{44}$$

Using expressions (31)–(36), we change equations (43) and (44) into

$$\lambda^* = \frac{\langle\lambda\nabla_x F_0\rangle}{\langle\nabla_x F_0\rangle} \tag{45}$$

$$\mu^* = \frac{\varepsilon\langle\lambda\nabla_x F_1\rangle + \langle\mu(\nabla F_0)^2\nabla_x F_0\rangle - \lambda^*\varepsilon\langle\nabla_x F_1\rangle}{\langle(\nabla F_0)^2\nabla_x F_0\rangle}. \tag{46}$$

When contact resistance exists between two different phases the temperature field is no longer continuous, $\nabla_x T_0$ and $\nabla_x T_1$ are singular at the interface. Therefore, the averages such as $\langle \nabla_x F_0 \rangle$, $\langle \nabla_x F_1 \rangle$ and $\langle (\nabla F_0)^2 \nabla_x F_0 \rangle$ are not only calculated using integrals over domains Ω_i and Ω_m, but should also include integrals over the interface. These interface integrals originate from the singularity of temperature gradients:

$$\langle \nabla_x F_0 \rangle = \langle \cdots \rangle_i + \langle \cdots \rangle_m + \langle \cdots \rangle_s \quad (47)$$

$$\langle \nabla_x F_1 \rangle = \langle \cdots \rangle_i + \langle \cdots \rangle_m + \langle \cdots \rangle_s \quad (48)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle = \langle \cdots \rangle_i + \langle \cdots \rangle_m + \langle \cdots \rangle_s \quad (49)$$

where, for convenience, the symbol ... is used to represent the integral functions on the left side. Subscripts i and m indicate integral domains Ω_i and Ω_m and subscript s refers to the interface. After careful analysis, we find the interface integrals:

$$\langle \nabla_x F_0 \rangle_s = \frac{kc}{BI} f \quad (50)$$

$$\langle \nabla_x F_1 \rangle_s = \frac{Kb_3}{BI} f \quad (51)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_s = [(1+B)^3 - c^3] f \quad (52)$$

where f is the volume concentration of inclusions.

Before deriving formulae of effective thermal conductivities, it is necessary to discuss several extreme cases to check the correctness of the above three expressions (50)–(52). In what follows, superscripts a, b and c, outside angle brackets, stand for the corresponding cases.

Case (a). Ideal contact. In this case, $h = \infty$, that is, $BI = \infty$. Temperatures are continuous and singularities of temperature gradients vanish at interfaces. Results of this paper will be completely the same as those of electrostatic boundary-value problems. So

$$\langle \nabla_x F_0 \rangle_s^a = 0 \quad (53)$$

$$\langle \nabla_x F_1 \rangle_s^a = 0 \quad (54)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_s^a = 0. \quad (55)$$

These results are as expected.

Case (b). Insulating contact. That is, $h = 0$ or $BI = 0$. Inclusions are completely isolated by infinite thermal contact resistance. Temperatures at the interfaces are no longer continuous. Temperatures in the inclusions are constant, being determined by their thermal history. Heat flow $q^i = 0$, therefore

$$\langle \nabla_x F_0 \rangle_s^b = 2f \quad (56)$$

$$\langle \nabla_x F_1 \rangle_s^b = -\frac{4}{3\lambda_m} \quad (57)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_s^b = 8f \quad (58)$$

$$\langle \nabla_x F_0 \rangle_i^b = 0 \quad (59)$$

$$\langle \nabla_x F_1 \rangle_i^b = 0 \quad (60)$$

$$\langle (\nabla F)^2 \nabla_x F_0 \rangle_i^b = 0. \quad (61)$$

Case (c). Insulating inclusions. In this case, inclusions are thermal insulators, that is, $\lambda_i = \mu_i = 0$. Temperatures at interfaces are continuous and the interface integrals become zero.

$$\langle \nabla_x F_0 \rangle_s^c = 0 \quad (62)$$

$$\langle \nabla_x F_1 \rangle_s^c = 0 \quad (63)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_s^c = 0 \quad (64)$$

$$\langle \nabla_x F_0 \rangle_i^c = 2f \quad (65)$$

$$\langle \nabla_x F_1 \rangle_i^c = -\frac{4}{3\lambda_m} \quad (66)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_i^c = 8f. \quad (67)$$

Though temperature gradients exist in the inclusions in this case, heat flow q^i is still equal to zero because $\lambda_i = \mu_i = 0$.

Comparison between case (b) and case (c) reveals that

$$\langle \nabla_x F_0 \rangle_s^b = \langle \nabla_x F_0 \rangle_i^c \quad (68)$$

$$\langle \nabla_x F_1 \rangle_s^b = \langle \nabla_x F_1 \rangle_i^c \quad (69)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_s^b = \langle (\nabla F_0)^2 \nabla_x F_0 \rangle_i^c. \quad (70)$$

In addition, it is very easy to prove that temperature distributions in the matrix of case (b) are completely the same as those of case (c), that is:

$$\langle \nabla_x F_0 \rangle_m^b = \langle \nabla_x F_0 \rangle_m^c \quad (71)$$

$$\langle \nabla_x F_1 \rangle_m^b = \langle \nabla_x F_1 \rangle_m^c \quad (72)$$

$$\langle (\nabla F_0)^2 \nabla_x F_0 \rangle_m^b = \langle (\nabla F_0)^2 \nabla_x F_0 \rangle_m^c. \quad (73)$$

This suggests that both case (b) and case (c) have the same effective conductivities. This conclusion is reasonable from the physical point of view. Composites in the two cases should share the same macroscopic heat transfer properties.

The above discussions show that the interface integrals yield reasonable results in every extreme case. After performing the other integrals in expressions (45) and (46), taking into account the low concentration of inclusions, we obtain

$$\frac{\lambda^*}{\lambda_m} = 1 + c(k - 1 - k/BI) f \quad (74)$$

$$\frac{\mu^*}{\mu_m} = 1 + \left[k_\mu c^3 - (1+B)^3 + (1 - 2B - B^3/3 - k_\mu c^3) \frac{k - 1 - k/BI}{k + 1 + k/BI} \right] f \quad (75)$$

where $k_\mu = \mu_i/\mu_m$.

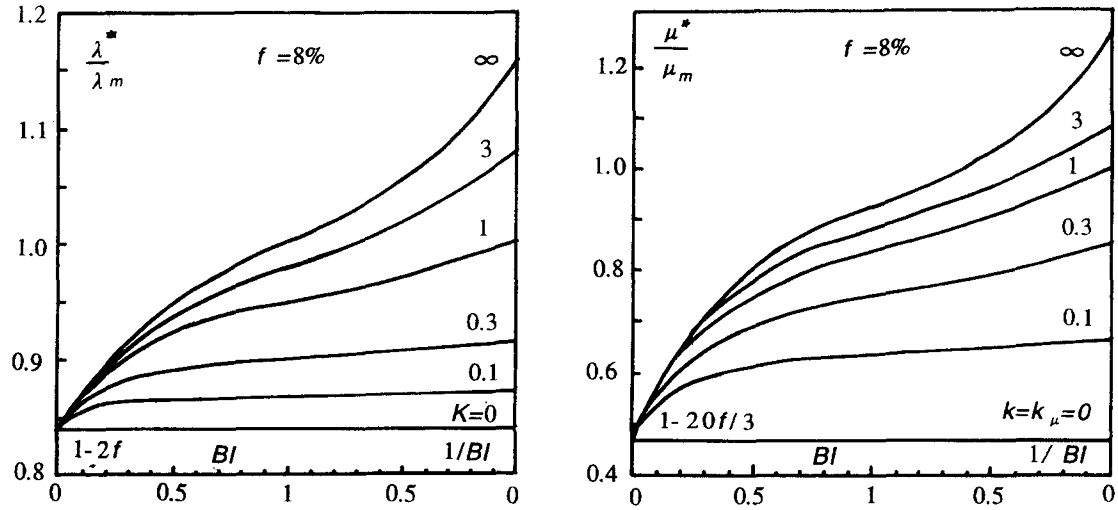

Fig. 1. Dependence of the effective thermal conductivities on BI. Values on curves for the first effective thermal conductivity (λ^*/λ_m) represent k and those on curves for the third effective thermal conductivity (μ^*/μ_m) represent k and k_μ (in this case, $k = k_\mu$).

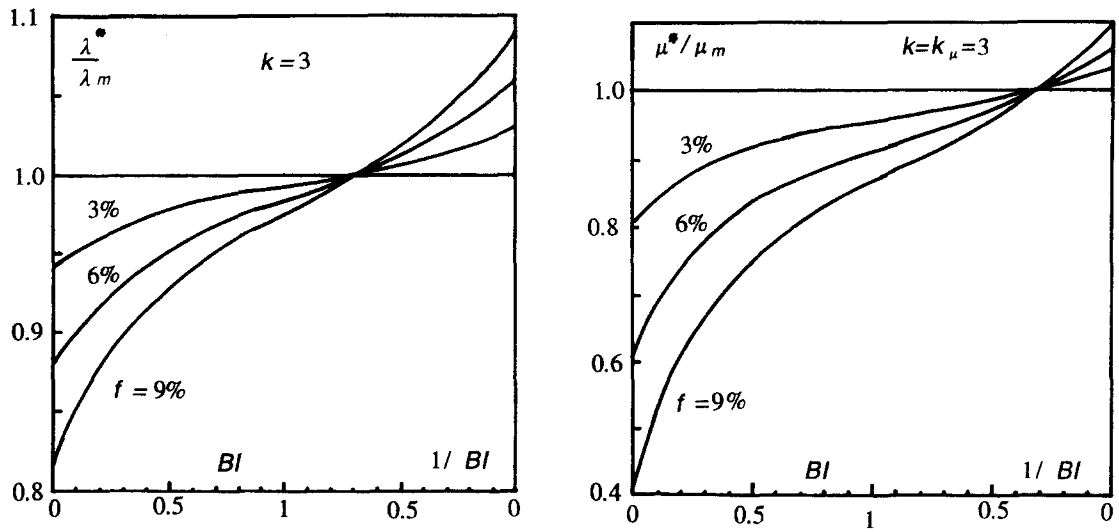

Fig. 2. Influence of the inclusion concentrations on effective thermal conductivities. Values on curves represent inclusion concentrations.

5. CONCLUSION

Using formulae (74) and (75), we study the influence of contact resistance on the thermal conductivity of nonlinear composite media. The dependence of first and third effective conductivities on Biot number BI are plotted in Fig. 1. They reflect the main features of systems with contact resistance. The effective conductivities are monotone increasing functions for decreasing contact resistance $1/BI$. The ratio of $k_\mu = \mu_i/\mu_m$ does not affect the effective conductivity λ^*/λ_m, however, the ratio of $k_\mu = \lambda_i/\lambda_m$ will influence both λ^*/λ_m and μ^*/μ_m. All curves with different k_i and k_μ converge at $BI = 0$, where $\lambda^*/\lambda_m = 1\text{–}2f$ and $\mu^*/\mu_m = 1\text{–}20f/3$. This reveals that no matter what conduction properties the inclusions have, they can be viewed as thermal insulators if they are completely isolated by infinite contact resistance. The effect of inclusion concentrations on the effective conductivities is depicted in Fig. 2.

Based on this research we can conclude that the analytical formulae (74) and (75) provide reasonable effective thermal conductivities of nonlinear composite media over all possible ranges of Biot number and that the contact resistance can change the effective conductivities greatly.

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